A LEVEL SET METHOD FOR THE COMPUTATION OF MULTIVALUED SOLUTIONS TO QUASI-LINEAR HYPERBOLIC PDES AND HAMILTON-JACOBI EQUATIONS

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Abstract. We develop a level set method for the computation of multivalued solutions to quasi-linear hyperbolic partial differential equations and the gradient flow of the Hamilton-Jacobi equations in any number of space dimensions. We use the classic idea of Courant and Hilbert to define the solution of the quasi-linear hyperbolic PDEs or the gradient of the solution to the Hamilton-Jacobi equations as zero level sets of level set functions. Then the evolution equations for the level set functions satisfy *linear Liouville equations* defined in the "phase" space, unfolding the singularities and preventing the numerical capturing of the viscosity solution. This provides a computational framework for the computation of multivalued geometric solutions to general quasilinear PDEs. By using the local level set method the cost of each time update for this method is $O(N^d \log N)$ for a d dimensional problem, where N is the number of grid points in each dimension.

Key words. Hyperbolic PDEs, Hamilton-Jacobi equations, level set method, multivalued solution, Liouville equation

AMS subject classifications.

1. Introduction.

Many physical problems require the computation of multivalued solutions. Such problems arise in the computation of dispersive waves [44, 27, 15], geometric optics [11, 33] , semiclassical linear and nonlinear Schrödinger equations [16, 25, 24, 39], multiple arrival in tomography and seismic migration [14, 41, 43], electron beam modulation in vacuum electronic devices (such as Klystron) [21, 28], etc. Although these problems are often, in the continuum limit, described by nonlinear hyperbolic partial differential equations or Hamilton-Jacobi equations, the classical entropy or viscosity solutions are not adequate in describing the post singularity behavior. The underlying physics is often defined by Hamiltonian systems which are nondissipative, often dispersive, and whose solution can be superimposed in the phase space. Direct numerical discretizations of these nonlinear PDEs often yield viscosity solutions which are irreversible, thus violating underlying physical principles, such as superposition.

There are two classes of methods used to compute multivalued solutions. A classical technique is ray tracing, which is a Lagrangian method that solves a set of ordinary differential equations in order to trace the wavefronts. This method is easy to implement, but may encounter difficulties in spatial resolution when points close initially may diverge at later times. This results in the loss of numerical accuracy unless a regridding and/or interpolation procedure is implemented from time to time. Another class involves Eulerian methods, which solve partial differential equations on a fixed grid, see e.g. [2],[3]. The new difficulty lies in the fact that a solution becomes multivalued in the physical space beyond singularity (caustic) formation.

In recent years, there has developed a growing interest in the computation of multivalued solutions of these nonlinear PDEs using Eulerian methods. Methods based in physical space often use moment closure of the classical Liouville equations. This approach started with the work of Brenier and colleagues [4, 5] and has been

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studied numerically by Engquist and Runborg [11] and Gosse [19] in the context of geometric optics. For linear Schrödinger equations multivalued solutions were studied in Sparber, Markowich and Mauser [39] and Jin and Li [25], and numerical studies of the resulting moment systems were studied in Jin and Li [25], Gosse, Jin and Li [20]. Computations based on moment systems are efficient when the number of phases is low in one space dimension. They become complex when the number of phases becomes large, and in multidimensions. For the computation of wavefronts, a Liouville equation based phase space techniques, using the segmentation projection method, was introduced by Engquist, Runborg and Tornberg [12], or using the level set method, was introduced by Osher et. al [33], [37], [9]. For computation of multiple arrivals using the static Hamilton-Jacobi equation, a Liouville equation based method was also introduced by Fomel and Sethian [14]. In all these newly developed Liouville based methods the goal was to compute the *wave front* instead of the solution in the entire physical domain.

In this paper we present a generic level set method for multidimensional quasilinear scalar hyperbolic PDEs and the gradient system of Hamilton-Jacobi equations that allows the computation of the solution in the *entire* physical domain. The basic idea for hyperbolic PDEs can be traced back to Jacobi [7], and Courant and Hilbert [10], and was recycled by Osher [32], then Evans, [13], Giga, [17], Giga and Sato [18], and Tsai, Giga and Osher [42] for the computation of viscosity solutions and generalizations. These earlier works focused on preventing multivalued solutions, while the emphasis of this paper is to use this formulation as a general framework for the computation of (geometric) multivalued solutions. The theoretical basis can be found in the classic book by Courant and Hilbert [10], where the solution (and its gradients) to quasilinear PDEs were built into the zero level set of level set functions. We provide an alternate derivation for general Hamilton-Jacobi equations, where the Hamiltonian depends only on the gradient of the solution, using the *gradient* of the Hamilton-Jacobi equation. This allows one to obtain a linear Liouville equation (with variable coefficients) in what resembles phase space. Due to the linearity of the level set Liouvile equation, the singularities of nonlinear PDEs are unfolded and linear superposition is preserved in this "phase" space. The intersection of the zero level sets of d such level set functions in d-space dimensions yield the desired multivalued solutions in the physical space. This follows from the vector level set method introduced in [6] and used in [33], [37], [9]. Both time dependent and static Hamilton-Jacobi equations can be treated in this fashion. This formulation is valuable if the quantities of interest are physical observables, which include the gradient of the phase, namely, velocity in the classical limit of the Schrödinger equation. When the Hamiltonian depends on both the solution and its gradient, or if one wants to construct the phase, a general level set equation given in [10] in an even higher dimension, where both the solution and its gradient are independent variables, can be used.

Although the level set equation is defined in the "phase" space, a local level set method can be used [1], [36] since the area of interest is near the zero level set. When the computation is restricted to the region near the zero level set, one can utilize a fast level set method whose computational cost is $O(N^d \log N)$ in d-space dimensions where N is the number of grid points in each space dimension. To our knowledge, this is the first time the multivalued surfaces were computed numerically for both quasilinear hyperbolic PDEs and multidimensional Hamilton-Jacobi equations. For quasilinear hyperbolic equations or more general Hamilton-Jacobi equations whose Hamiltonians grow no faster than $O(|\nabla u|)$ as $|\nabla u| \to \infty$, level set formulation of this

type was used to generate viscosity solutuons [42], where the goal was to prevent the formation of multi-valued solution.

In Osher, et al [33],[37],[9] the authors were concerned with following a curve in $\Re^2$ or a surface in $\Re^3$ as each point evolves according to the bicharacteristics of the associated Hamiltonian. This is implemented by viewing the set $\Gamma(t) \in \Re^d$ as the common zero level set of a functions of 2d - 1 variables. Each update of the Liouville equation involves therefore $O(N^{d-1} \log N)$ for the eikonal equation and $O(N^d \log N)$ operations for general Hamiltonians.

In Fomel and Sethian [14] the authors compute *all* arrival times for a set of codimension one in $\Re^d$. One level set function is used in a static setting. Thus each update involves $O(N^{2d-1} \log N)$ operations for the eikonal equation.

In our present work we take an entire region of zero codimension in $\Re^2$ or $\Re^3$ (or $\Re^d$) and update it according to the bicharacteristic flow, i.e. we solve an initial value problem for the multivalued solution with data given on a subset of $\Re^d$. We use d level set functions of 2d variables. Thus every update has complexity $O(N^d \log N)$ in general.

After the completion of this work we were informed by Hailiang Liu that the same formulation from Courant and Hilbert was also utilized as a general framework to compute multivalued solutions to first order PDEs in their ongoing work [29]. This idea was also mentioned in a recent work by Cheng, Liu and Osher [8], which was motivated by high frequency wave propagation for Schrödinger's equation. This work also involved the development of a procedure for Hamilton-Jacobi equations, including the computation of the phase.

Compared with the moment approach, the level set method has the advantage of being able to obtain multivalued solutions regardless of the number of phases, while the moment system is mostly effective when that number is small and it becomes very complicated when the number of phases is large, and hard to construct in higher space dimensions. The dimensionality of the level set method is higher than the moment system, which is defined in the physical space, so the cost of the level set method based on direct discretization in the phase space could become prohibitively expensive unless a high codimensional local level set method is implemented. This is being done in [31]. So far, for multivalued solutions, the level set method has only been constructed for scalar problems, while the moment method is applicable to more generic nonlinear systems such as the Euler-Poisson system [28].

2. A Level Set Equation for Multivalued solutions.

In this section we review the classic results of Jacobi, and Courant and Hilbert, for the derivations of the level set equations for multidimensional hyperbolic PDEs. To generalize the idea to the Hamilton-Jacobi equation where the Hamiltonian does not depend on the solution, only its gradient, we give an alternate procedure for obtaining the Liouville equation for level set functions by handling the *gradient* of the Hamilton-Jacobi equation. This gives a way to construct multivalued solution of *velocity*, the gradient of the phase, in a space with dimension twice as big as the original physical problem. A more general procedure from Courant and Hilbert [10], which uses both the solution and its gradient as independent variables, yields linear level set equations in even higher dimension but allows one to compute both the phase and velocity.

Consider $\mathbf{x} = (x_1, x_2, \cdots x_d) \in \Re^d$.

2.1. Multidimensional hyperbolic PDEs. Let $u(t, \mathbf{x}) \in \Re$ be a scalar satisfying an initial value problem of an d-dimensional first order hyperbolic PDE with

source term:

$$\partial_t u + \mathbf{F}(u) \cdot \nabla_{\mathbf{x}} u + q(\mathbf{x}, u) = 0\,, \tag{2.1}$$

$$u(0, \mathbf{x}) = u_0(\mathbf{x})\,. \tag{2.2}$$

Here $\mathbf{F}(u) : \Re \to \Re^d$ is a vector, and $q : \Re^{d+1} \to \Re$ is the source term. This equation includes any such hyperbolic PDE in conservative or nonconservative form. Introduce a level set function $\phi(t, \mathbf{x}, p)$ in dimension $d+1$ plus time, whose zero level set is the solution u:

$$\phi(t, \mathbf{x}, p) = 0 \qquad \text{at} \quad p = u(t, \mathbf{x})\,. \tag{2.3}$$

Therefore one evolves the entire solution u as the zero level set of ϕ.

Taking the time derivative of (2.3), one obtains

$$\partial_t \phi + \partial_p \phi\, \partial_t p = 0\,, \tag{2.4}$$

or, by using the equation (2.1) and $p = u$,

$$\partial_t \phi - \partial_p \phi\, \mathbf{F} \cdot \nabla_{\mathbf{x},p} u - q(\mathbf{x}, p) \partial_p \phi = 0\,. \tag{2.5}$$

Taking the gradient in $\mathbf{x}$ of (2.3) one has

$$\nabla_{\mathbf{x}} \phi + \partial_p \phi\, \nabla_{\mathbf{x}} u = 0\,. \tag{2.6}$$

Applying (2.6) in (2.5), one obtains

$$\partial_t \phi + \mathbf{F}(p) \cdot \nabla_{\mathbf{x}} \phi - q(\mathbf{x}, p)\, \partial_p \phi = 0\,. \tag{2.7}$$

This is the level set equation. It resembles a Liouville equation, which is linear hyperbolic with variable coefficients, in $d+1$ space variables. Of course $x \in \Re^d$ and $p \in \Re^1$, so this is not a true Liouville equation, for $d > 1$. For the Hamilton-Jacobi equation analyzed below, we will arrive at a true Liouville equation. However, even there, p does not denote phase in the classical sense. Instead it is just the auxiliary set of variables needed to generate the level set motion.

For smooth initial data $u_0(x)$, the initial condition for ϕ can be chosen simply as

$$\phi(0, \mathbf{x}, p) = p - u_0(\mathbf{x})\,. \tag{2.8}$$

However, if the initial data is discontinuous, such as in the Riemann problem, such a choice of the initial level set will miss the line that connects the two constant states, thus forming a vacuum, as can be seen from the exact solution of the Liouville equation which can be solved analytically in one space dimension using the method of characteristics. In this case, a good choice for initial level set function is the *signed distance function* to the interface $p = u_0(\mathbf{x})$, which is continuous and will remain continuous for all time due to the linearity of the Liouville equation. Thus this level set method always selects the *continuous* multi-valued solution. This distance function can be approximated by numerically solving $|\nabla d| = 1$, with $d = 0$ when (a) $p = u(\mathbf{x})$ and (b) along the vertical hypersurface connecting the discontinuity in $(\mathbf{x}, p)$ space, with d changing sign across this interface. See [40] and [42].

2.2. Time dependent Hamilton-Jacobi equations. Consider the time dependent, d-dimensional Hamilton-Jacobi equation

$$\partial_t S + H(\mathbf{x}, \nabla_{\mathbf{x}} S) = 0\,, \tag{2.9}$$

$$S(0, \mathbf{x}) = S_0(\mathbf{x})\,. \tag{2.10}$$

Introduce $\mathbf{u} = (u_1, \cdots, u_d) = \nabla_{\mathbf{x}} S$. Taking the gradient on (2.9), one gets an equivalent (at least for smooth solutions) form of the Hamilton-Jacobi equation

$$\partial_t \mathbf{u} + \nabla_{\mathbf{x}} H(\mathbf{x}, \mathbf{u}) = 0\,, \tag{2.11}$$

$$\mathbf{u}(0, \mathbf{x}) \equiv \mathbf{u}_0(\mathbf{x}) = \nabla_{\mathbf{x}} S_0(\mathbf{x})\,. \tag{2.12}$$

This conservation law formulation and its equivalence with the original Hamilton-Jacobi equation, in the sense of viscosity solution, was discussed in [26] and used to construct numerical schemes for the original Hamilton-Jacobi equation. It can also be used to construct the discontinuous Galerkin method for Hamilton-Jacobi equations [22]. This system is an example of what was called systems with the same principle part in [10]. A key condition to guarantee the condition is that $\mathbf{u}$ *remains a gradient*:

$$\mathbf{u} = \nabla_{\mathbf{x}} S\,, \tag{2.13}$$

which ensures that

$$\nabla_{\mathbf{x}} \mathbf{u} = \nabla_{\mathbf{x}}^2 S\,, \tag{2.14}$$

the Hessian matrix of S, is *symmetric*, namely,

$$(\nabla_{\mathbf{x}} \mathbf{u})^T = \nabla_{\mathbf{x}} \mathbf{u}\,. \tag{2.15}$$

As will be seen later, this is a critical condition for our derivation of the level set equation.

We use d level set functions $\phi_i = \phi_i(t, \mathbf{x}, \mathbf{p}),\ i = 1, \cdots, d$, where $\mathbf{p} = (p_1, \cdots, p_d) \in \Re^d$, such that the intersection of their zero level sets yields $\mathbf{u}$, namely,

$$\phi_i(t, \mathbf{x}, \mathbf{p}) = 0 \qquad \text{at} \quad \mathbf{p} = \mathbf{u}(t, \mathbf{x})\,, \qquad i = 1, \cdots, d \tag{2.16}$$

We now derive the equation for ϕ_i $(i = 1, \cdots, d)$ as in the preceding subsection.

Taking the time derivative of (2.16), one gets

$$\partial_t \phi_i + \nabla_{\mathbf{p}} \phi_i \cdot \partial_t \mathbf{p} = 0 \tag{2.17}$$

or, by using equation (2.11) and $\mathbf{p} = \mathbf{u}$,

$$\partial_t \phi_i - \nabla_{\mathbf{p}} \phi_i \cdot \nabla_{\mathbf{x}} H(\mathbf{x}, \mathbf{p}) = 0\,. \tag{2.18}$$

Using $\nabla_{\mathbf{x}} H(\mathbf{x}, \mathbf{p}) = \nabla_{\mathbf{x}} \mathbf{u} \nabla_{\mathbf{p}} H + \nabla_{\mathbf{x}} H$ in (2.18) one gets

$$\partial_t \phi_i - \nabla_{\mathbf{p}} \phi_i \cdot \nabla_{\mathbf{x}} \mathbf{u} \nabla_{\mathbf{p}} H - \nabla_{\mathbf{p}} \phi_i \cdot \nabla_{\mathbf{x}} H = 0\,. \tag{2.19}$$

Taking the gradient in $\mathbf{x}$ of (2.16) one has

$$\nabla_{\mathbf{x}} \phi_i + \nabla_{\mathbf{x}} \mathbf{u} \nabla_{\mathbf{p}} \phi_i = 0\,. \tag{2.20}$$

Note

$$\begin{aligned}\nabla_{\mathbf{p}}\phi_i \cdot \nabla_{\mathbf{x}}\mathbf{u}\nabla_{\mathbf{p}}H &= (\nabla_{\mathbf{p}}H)^T(\nabla_{\mathbf{x}}\mathbf{u})^T\nabla_{\mathbf{p}}\phi_i = (\nabla_{\mathbf{p}}H)^T\nabla_{\mathbf{x}}\mathbf{u}\nabla_{\mathbf{p}}\phi_i \\ &= -(\nabla_{\mathbf{p}}H)^T\nabla_{\mathbf{x}}\phi_i\end{aligned}$$

where in the second equality we used the symmetry (2.14) while in the third equality we used (2.20). Thus ϕ_i solves the following equation:

$$\partial_t\phi_i + \nabla_{\mathbf{p}}H \cdot \nabla_{\mathbf{x}}\phi_i - \nabla_{\mathbf{x}}H \cdot \nabla_{\mathbf{p}}\phi_i = 0\,, \quad i = 1, \cdots, d\,. \tag{2.21}$$

It is the Liouville equation, which is linear hyperbolic with variable coefficients since in (2.21) $H = H(\mathbf{x}, \mathbf{p})$. A convenient initial condition for each $\phi_i,\ i = 1, \ldots, d$ can be taken as:

$$\phi_i(0, \mathbf{x}, \mathbf{p}) = p_i - u_i(0, \mathbf{x})\,, \tag{2.22}$$

if $\mathbf{u}_0(\mathbf{x})$ is continuous. Otherwise we choose $\phi_i(0, \mathbf{x}, \mathbf{p})$ to be the signed distance to the interface $p_i = u_i(\mathbf{x})$. We can treat discontinuous gradients of $S_0(x)$ this way, as we described in the previous section.

To visualize the solution which is an intersection of the zeroes of several level set functions one may use the technique developed in [30].

One can obtain the exact solution of the level set equations (2.21) using the bicharacteristics. Introduce the Hamiltonian flow,

$$\begin{cases} \mathbf{x}'(t) &= \nabla_{\mathbf{p}}H(\mathbf{x}, \mathbf{p})\,, \qquad \mathbf{x}(0) = \mathbf{x}_0, \\ \mathbf{p}'(t) &= -\nabla_{\mathbf{x}}H(\mathbf{x}, \mathbf{p})\,; \qquad \mathbf{p}(0) = \mathbf{p}_0. \end{cases} \tag{2.23}$$

We denote the solution to these ODEs by $\mathbf{x} = \mathbf{x}(t, \mathbf{x}_0, \mathbf{p}_0)$ and $\mathbf{p} = \mathbf{p}(t, \mathbf{x}_0, \mathbf{p}_0)$. The curves $(\mathbf{x}, \mathbf{p})$ are the *bicharacteristics* in phase space. Assume H is smooth. Since the Hamiltonian flow is volume preserving, the bicharacteristic curves defined by (2.23) are invertible and we denote the inverse functions by $(\mathbf{x}_0(t, \mathbf{x}, \mathbf{p}), \mathbf{p}_0(t, \mathbf{x}, \mathbf{p}))$. Since the solution of the Liouville equation (2.21) is a constant along the bicharacteristic curve defined by (2.23), one has

$$\phi_i(t, \mathbf{x}, \mathbf{p}) = \phi_i(0, \mathbf{x}_0(t, \mathbf{x}, \mathbf{p}), \mathbf{p}_0(t, \mathbf{x}, \mathbf{p})). \tag{2.24}$$

Therefore,

$$(\phi_1, \cdots, \phi_d)^T = \mathbf{p}_0(t, \mathbf{x}, \mathbf{p}) - \mathbf{u}_0(\mathbf{x}_0(t, \mathbf{x}, \mathbf{p}))\,. \tag{2.25}$$

2.3. Static Hamilton-Jacobi equations. We now derive the level set equations for the d-dimensional static Hamilton-Jacobi equation

$$H(\mathbf{x}, \nabla_{\mathbf{x}}S) = 0\,. \tag{2.26}$$

Introduce $\mathbf{u} = \nabla_{\mathbf{x}}S$. Taking the gradient on (2.26), one gets an equivalent (at least for smooth solutions) form of the Hamilton-Jacobi equation

$$\nabla_{\mathbf{x}}H(\mathbf{x}, \mathbf{u}) = 0\,, \tag{2.27}$$

Again we need that condition

$$\mathbf{u} = \nabla_{\mathbf{x}}S\,, \tag{2.28}$$

which ensures that symmetry of $\nabla_{\mathbf{x}}\mathbf{u} = \nabla^2_{\mathbf{x}} S$.

We use d level set functions $\phi_i = \phi_i(t, \mathbf{x}, \mathbf{p})$ $(i = 1, \cdots, d)$ such that the intersection of their zero level sets yields $\mathbf{u}$, namely,

$$\phi_i(\mathbf{x}, \mathbf{p}) = 0 \qquad \text{at} \quad \mathbf{p} = \mathbf{u}(t, \mathbf{x}) \,, \qquad i = 1, \cdots, d \,. \tag{2.29}$$

We now derive the equation for ϕ_i $(i = 1, \cdots d)$.

Taking the gradient in $\mathbf{x}$ of (2.29) one has

$$\nabla_{\mathbf{x}}\phi_i + \nabla_{\mathbf{x}}\mathbf{u}\,\nabla_{\mathbf{p}}\phi_i = 0 \,. \tag{2.30}$$

Taking gradient with respect to $\mathbf{x}$ on equation $H(\mathbf{x}, \mathbf{u}) = 0$ one gets

$$\nabla_{\mathbf{x}} H + \nabla_{\mathbf{x}}\mathbf{u}\,\nabla_{\mathbf{p}} H = 0 \,. \tag{2.31}$$

Note

$$\begin{aligned} \nabla_{\mathbf{p}}\phi_i \cdot \nabla_{\mathbf{x}}\mathbf{u}\,\nabla_{\mathbf{p}} H \qquad & = (\nabla_{\mathbf{p}} H)^T (\nabla_{\mathbf{x}}\mathbf{u})^T \nabla_{\mathbf{p}}\phi_i = (\nabla_{\mathbf{p}} H)^T \nabla_{\mathbf{x}}\mathbf{u} \nabla_{\mathbf{p}}\phi_i \\ & = -(\nabla_{\mathbf{p}} H)^T \nabla_{\mathbf{x}}\phi_i \end{aligned} \tag{2.32}$$

where in the second equality we used the symmetry of $\nabla_{\mathbf{x}}\mathbf{u}$ while in the third equality we used (2.30). Multiplying (2.31) by $\nabla_{\mathbf{p}}\phi_i$ and using (2.32) give the following static Liouville equation

$$\nabla_{\mathbf{p}} H \cdot \nabla_{\mathbf{x}}\phi_i - \nabla_{\mathbf{x}} H \cdot \nabla_{\mathbf{p}}\phi_i = 0 \,. \tag{2.33}$$

2.4. Computation of the phase. If one is interested in computing the phase S, or if the Hamiltonian depends also on the solution in addition to its gradient, one can simply add one more dimension in the level set function and the rest of the calculation is similar, and can be found in [10]. This was also done in [8], [29].

3. Numerical Discretization.

We need to solve the system (2.21) of d linear equations in 2d unknowns, plus time. This is a decoupled system of linear equations with variable coefficients. Thus we need only consider linear equations of the form

$$\partial_t \psi + \mathbf{a} \cdot \nabla_{\mathbf{x}}\psi + \mathbf{b} \cdot \nabla_{\mathbf{p}}\psi = 0, \mathbf{x} = (x_1, \ldots, x_d), \mathbf{p} = (p_1, \ldots, p_d)$$

We shall use the ENO and WENO schemes for Hamilton-Jacobi equations developed in [35],[23], respectively as well as the TVD Runge-Kutta time discretizations of Shu-Osher [38]. We are solving a linear Hamilton-Jacobi equation so the numerical Hamiltonian can be taken to be simple (high order ENO or WENO) upwind difference. We use a uniform grid in each x_i with grid spacing Δx_i, and in p_i with spacing Δp_i. For details see [35],[23] or a complete overview in the recent book [34].

One can also use a local level set method as in [33],[37],[9], [8] originating in [36]. This lowers the complexity of each update down to $O(N^d) \log N$ where N is the number of points in each direction and N^d represents the number of points on the zero level set of the vector functions $\psi(\mathbf{x}, t)$. A similar problem was treated numerically in [33] with a somewhat detailed exposition. A technique recently developed in [31] also reduces the storage requirements to $O(N^d \log N)$ as well. We shall use this technique in future work.

4. Numerical Examples.

In this section, we present several numerical examples.

4.1. One-dimensional hyperbolic equations. We use the Burgers' equation with forcing as an example. This equation arises as the semiclassical limit of the linear Schrödinger equation.

$$\partial_t u + u\partial_x u + \partial_x V = 0\,, \tag{4.1}$$

$$u(0,x) = u_0(x)\,, \tag{4.2}$$

where $V(x)$ is the potential.

Example 4.1: A free particle model for a Gaussian pulse. We first consider the following problem which models the evolution in time of a pulse-type signal for (4.1) with zero potential $V(x) \equiv 0$. The initial condition for $x \in [-1,1]$ is

$$u_0(x) = -\sin(\pi x)|\sin(\pi x)|^{k-1}. \tag{4.3}$$

Intuitively, the decreasing interval of the velocity profile will eventually lead to caustics. The examples considered here have been given in [19, 20].

The mesh sizes are $\Delta x = \Delta p = 0.01$ for all the runs. The time step is chosen to satisfy the CFL condition. The numerical results are shown at time $t = 0$ and $t = 1$. Periodic boundary conditions are used in the computational domain $[-1,1] \times [-1.5, 1.5]$. We use a larger domain in p to avoid dealing with boundary conditions in the p direction. This is not physical, but due to finite propagation speed it will not affect the solution in the interior of the time up to the time of our computation. We use third order ENO method and a second order TVD Runge-Kutta method for time discretization. The numerical results are shown in Fig. 4.1.

One cusp caustic: k=1. In this case, the exact solution of the rays contains self-interference, develops a cusp singularity, and hence admits at most three phases.

Two cusps merging into one: k=2. In this case, the initial data u_0 has an inflection point at $x = 0$ and two cusp singularities develop inside the geometric solution which exhibits at most five phases.

The 'square signal': k=100. In this case, the initial data develops two cusp singularities involving five phases for all later time.

Example 4.2: Discontinuous initial data. We consider two Riemann problems. The first one is given by

$$u_0(x) = \begin{cases} 1, & x < 0; \\ 0, & x > 0. \end{cases} \tag{4.4}$$

For viscosity solution this initial data yields a shock moving to the right with speed $1/2$. Periodic boundary condition is used in the computational domain $[-2,2] \times [-0.1, 1.1]$. The mesh size are $\Delta x = \Delta p = 0.01$. The time step is chosen to satisfy the CFL condition. We use a larger domain in p for the convenience of boundary condition. The numerical results at $t = 0.5$ are shown on the top of Fig. 4.2.

The second initial condition is chosen to be

$$u_0(x) = \begin{cases} -1, & x < 0; \\ 1, & x > 0. \end{cases} \tag{4.5}$$

This initial data yields a rarefaction wave travelling to the right with the characteristic speed. The exact solution is a line of slope x/t connecting -1 from the left to 1 from the right. Periodic boundary condition is used in a computational domain $[-2,2] \times$

$[-1.1, 1.1]$. The mesh size are $\Delta x = \Delta p = 0.01$, and time step is chosen to satisfy the CFL condition. We use a larger domain in p to avoid dealing with boundary conditions in the p direction. The numerical results $t = 0.5$ are shown in the bottom of Fig. 4.2.

In both cases the signed distance functions are used for the initial condition of the level set functions.

Example 4.3: The harmonic oscillator. Consider a confining potential

$$V(x) = \frac{1}{2}x^2,$$

together with the initial conditions:

$$u_0(x) = -\tanh(5x). \tag{4.6}$$

The exact solution can be found by the method of characteristics:

$$x(t) = \cos(t)x_0 + \sin(t)u_0(x_0), \qquad u(x(t), t) = -\sin(t)x_0 + \cos(t)u_0(x_0).$$

This corresponds to a rotation in the phase space at unitary angular speed.

The mesh size are $\Delta x = \Delta p = 0.01$. The time step is chosen to satisfy the CFL condition. The results are shown at time $t = 0.5$. Periodic boundary conditions are used in the computational domain $[-2, 2] \times [-2, 2]$. We use a larger domain in both x and p to avoid trouble with the boundary conditions. The numerical result is given in Fig.4.3.

In all above numerical examples the numerical solutions agree with those given in [20] using ray tracing.

4.2. A two-dimensional hyperbolic equation. As an example we use the two dimensional Burgers equation

$$\partial_t u + u\partial_x u + u\partial_y u = 0\,. \tag{4.7}$$

Example 4.4: A Riemann problem. Consider the initial data:

$$u_0(x, y) = \begin{cases} -1.0, & x > 0, y > 0; \\ -0.2, & x < 0, y > 0; \\ 0.5, & x < 0, y < 0; \\ 0, & x > 0, y < 0. \end{cases} \tag{4.8}$$

We compute the problem over domain $[-1, 1]^4$ with reflecting boundary conditions. 40 cells are used in each direction and $\Delta t = 0.025$. The numerical solution at $t = 0.8$ is shown in Fig. 4.4. One sees a multivalued surface.

Example 4.5: A continuous initial data Consider the following initial condition:

$$u_0(x, y) = 0.45 \cos \pi x[\sin(\pi y) - 1]\,. \tag{4.9}$$

We solve the problem over domain $[-1, 1] \times [-1, 1] \times [-1, 1]$ with periodic boundary conditions. 20 cells are used in each direction and $\Delta t = 0.025$. The numerical solution at $t = 1$ is show in Fig. 4.5.

4.3. A two-dimensional Hamilton-Jacobi equations. Consider the eikonal equation:

$$\partial_t S + \frac{1}{2}|\nabla S|^2 = 0\,, \qquad \mathbf{x} \in \Re^2\,. \tag{4.10}$$

This eikonal equation arises as the semiclassical limit of the linear Schrödinger equation, where S is the phase. It is valid before the formation of caustics. Beyond caustics, the velocity $\mathbf{u}(u,v) = \nabla S \in \Re^2$ becomes multivalued [25, 39]. It is the multivalued solution that we are interested in computing here.

Taking the gradient of (4.10), one obtains the momentum equations of pressureless gas equations for $\mathbf{u}$:

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = 0 \tag{4.11}$$

We solve two level set equations (2.21) for two level set functions with initial data given by (2.22), with the Hamiltonian $H(\mathbf{x},\mathbf{u}) = \frac{1}{2}|\mathbf{u}|^2$. The intersection of zero level hypersurfaces of the two level set functions yields the solution u and v.

Consider the following initial condition:

$$u_0(x,y) = 0.45\cos\pi x[\sin(\pi y) - 1]\,, \tag{4.12}$$

$$v_0(x,y) = 0.45\cos\pi y[\sin(\pi x) - 1]\,, \tag{4.13}$$

We solve the problem over a domain $[-1,1]^3$ with periodic boundary conditions. 20 cells are used in each direction and $\Delta t = 0.025$. The numerical solution at $t = 1$, is shown in Fig. 4.6, which not only has z-shaped multivalued solution but also a pipe-like surface crossing. The surface was plotted using the algorithm developed in [30].

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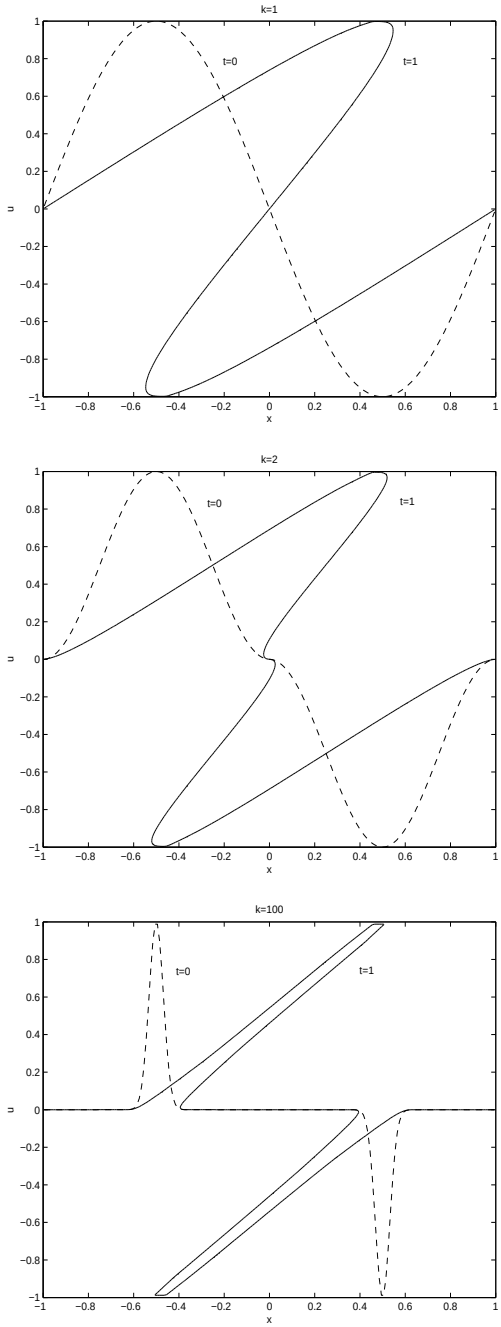


FIG. 4.1. *The contour plots of u for different k's with initial condition (4.3).*

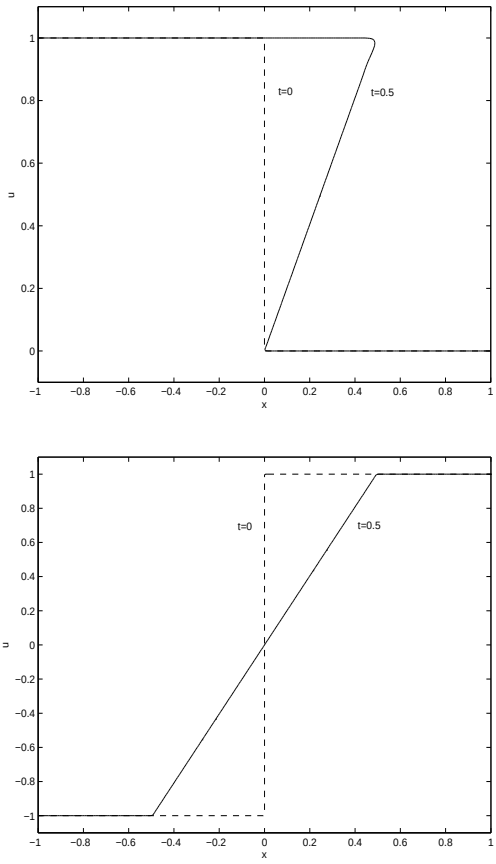


FIG. 4.2. *The contour plots of u for Riemann problem. Top: shock initial data (4.4); bottom: rarefaction initial data (4.5).*

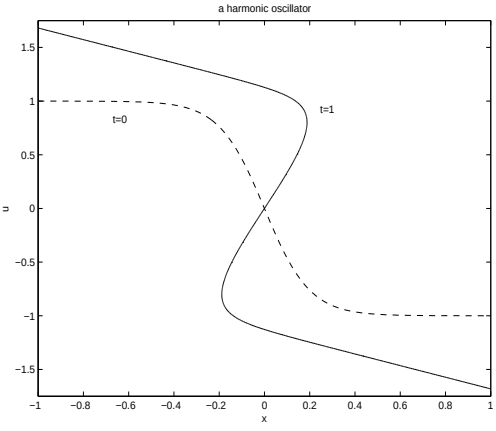


FIG. 4.3. *The numerical solutions of the harmonic oscillator case.*

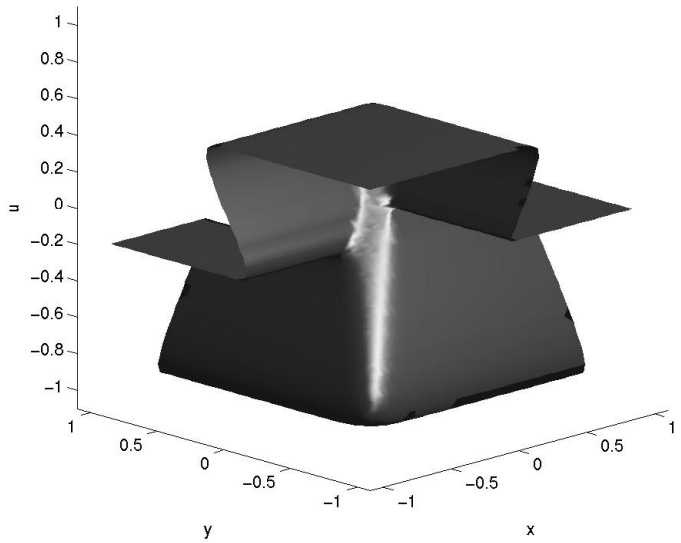


FIG. 4.4. *The numerical solutions of the two-d Burgers' equation (4.7) with initial data (4.8) at* $t = 0.8$.

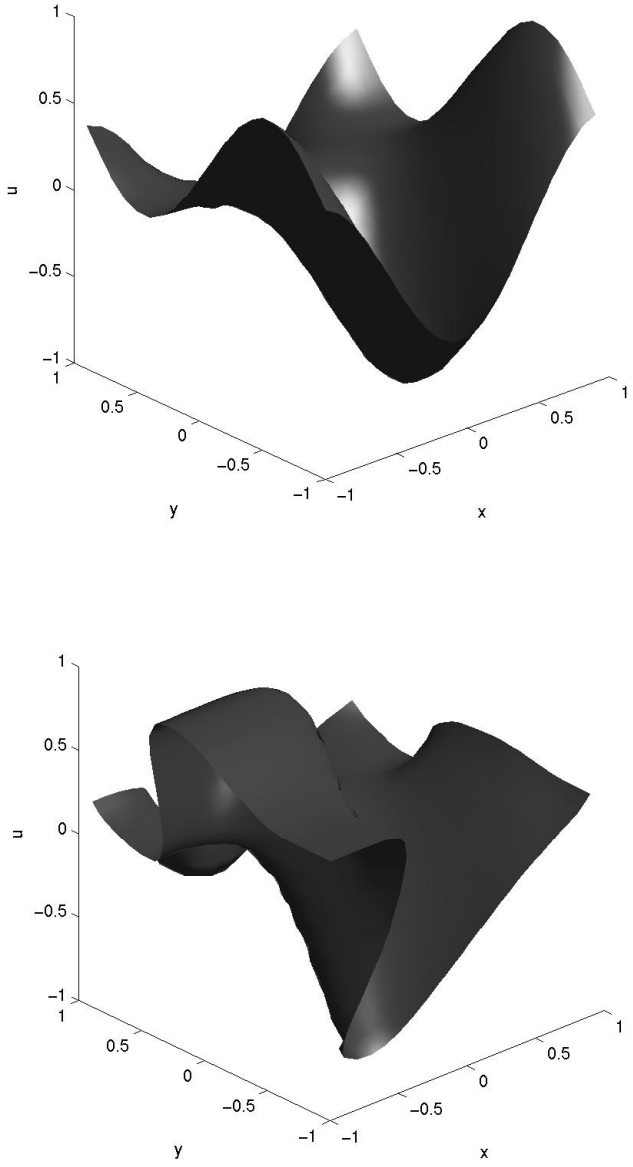
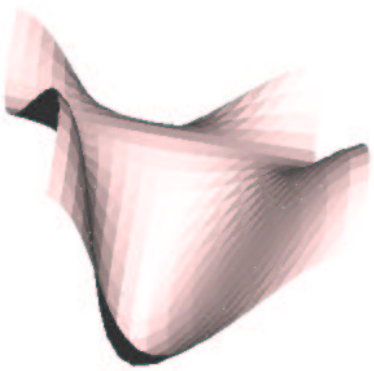
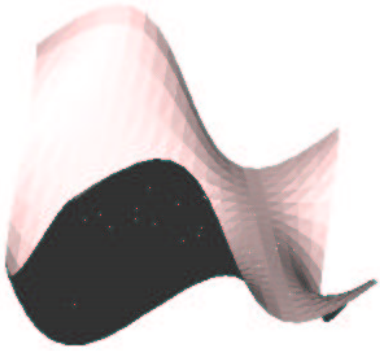


FIG. 4.5. *u for initial data (4.9). Top: initial data; bottom: solution at* $t = 1$.



(a)



(b)

(c)

(d)

FIG. 4.6. *Numerical solutions for the 2d pressureless gas equation (4.11). From top to bottom: (a) u at $t = 0$; (b) v at $t = 0$; (c) u at $t = 1$; (d) v at $t = 1$.*